General Relativity $\boxed{\text{Week 6}}$

Last time: We talked about some causality conditions on spacetimes (M, g)

- Causal: No closed causal curves
- Strongly causal: No almost closed causal curves
- Stably causal: $\exists$ function f with ∇f timelike everywhere
  $\Leftrightarrow$ causal under small perturbations of g

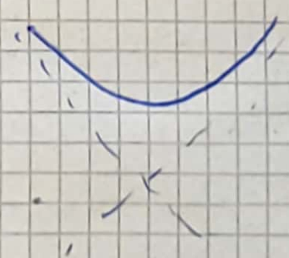
- Globally hyperbolic: Spacetime that admits a Cauchy hypersurface

Def: Cauchy hypersurface Σ: A hypersurface which is intersected by every inextendible causal curve exactly once

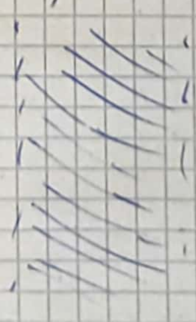
So Σ: cannot have timelike or null pieces.

e.g. In Minkowski $(\mathbb{R}^{n+1}, \eta)$:

- $\{t = \text{const}\}$ is a Cauchy hypersurface
- $-t^2 + \sum_{i=1}^{n} (x^i)^2 = \text{const}$ is not



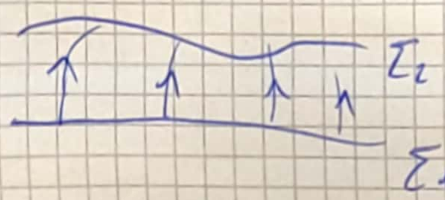
- The strip



$\subseteq (\mathbb{R}^{1+1}, \eta)$ is not globally hyperbolic

Note: If Σ_1 and Σ_2 are Cauchy hypersurfaces of (M,g) then they are homeomorphic

Proof: Fix a timelike vector field V (possible since (M,g) is a spacetime). Each flow line of V intersects Σ_1, Σ_2 exactly once, hence defines a bi-continuous map $\Sigma_1 \to \Sigma_2$ $\square$

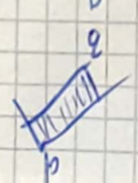


Prop: If (M,g) is globally hyperbolic and Σ is a Cauchy hypersurface, then
$$M \cong \mathbb{R} \times \Sigma$$

Sketch of proof: Fix a timelike vector field V and "suitably normalize" it so that each integral curve exists for infinite time.
Then the flow map $\phi^{(V)} : \mathbb{R} \times \Sigma \longrightarrow M$ provides this identification $\square$

Equivalent definitions (without proof - See Wald Ch. 8 or Hawking-Ellis)
(M,g) spacetime:

Globally hyperbolic $\Longleftrightarrow$ Strongly causal & $J^+(p) \cap J^-(q)$ compact $\forall p,q$ $\Longleftrightarrow$ Strongly causal & the space $C(p,q)$ of causal curves from p to q is compact in the C^0 topology.



Note: In Minkowski. If $\gamma \in C(p,q)$, it is a Lipschitz curve when parametrized by t
$\Rightarrow$ compactness follows by Arzela-Ascoli.

Global hyperbolicity: Lorentzian analogue of "completeness"

E.g. If $q \in J^+(p)$: $\exists$ causal geodesic connecting p to q, follows from compactness of $C(p,q)$ by maximizing the length

<u>And</u>: <u>Prop</u> If (M,g) is globally hyperbolic:
$\exists$ <u>smooth time function</u> $t: M \to \mathbb{R}$
(i.e. ∇t is timelike) such that $\Sigma_\tau = \{t = \tau\}$
is a Cauchy hypersurface $\forall \tau \in \mathbb{R}$

(So in particular: global hyperbolicity $\Rightarrow$ stable causality)

<u>Riemann Curvature tensor</u>

Let (M,g) be a Lorentzian manifold

$(1,3)$ tensor field defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

It is a tensor field! (Check that it is C^∞-multilinear)

Components: $R^\alpha{}_{\beta\gamma\delta} = \left(R(\partial_\gamma, \partial_\delta)\partial_\beta\right)^\alpha$

We will frequently lower the index: $R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda} R^\lambda{}_{\beta\gamma\delta}$

<u>Note</u>: In coordinates: $R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\lambda}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\delta\lambda}\Gamma^\lambda_{\beta\gamma}$

Also: $R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta}\right)$
$+ g \cdot \partial g \cdot \partial g$

<u>Note</u>: In normal coordinates: At 0, only the 2^{nd} order terms survive!

Symmetries of the Riemann tensor:

- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$
- $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$
- 1^{st} Bianchi identity: $R_{\alpha\beta\gamma\delta} + R_{\gamma\alpha\beta\delta} + R_{\beta\gamma\alpha\delta} = 0$
- 2^{nd} Bianchi identity: $\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} = 0$

thus: $(\nabla R)_{\alpha\beta\gamma\mu\nu}$

Sketch of proof: The above are tensorial identities, it suffices to show that they are true in some coordinate system.

Choosing normal coordinates at p:

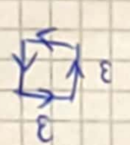
Use that $R_{\alpha\beta\gamma\delta}|_p = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\beta\delta} g_{\alpha\gamma} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\alpha\gamma} g_{\beta\delta}\right)\Big|_p$

and $\partial^2_{\alpha\beta} g_{\gamma\delta} + \partial^2_{\gamma\alpha} g_{\beta\delta} + \partial^2_{\beta\gamma} g_{\alpha\delta}\Big|_p = 0$

∎

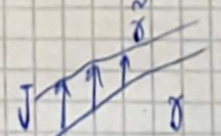
Physical interpretation:

- Parallel transport around a closed infinitesimal loop:



R measures the failure of that to be the identity

- Tidal forces in nearby geodesics (see Jacobi fields): $D_{\dot\gamma} D_{\dot\gamma} J + R(J,\dot\gamma)\dot\gamma = 0$



On Minkowski: $R_{\alpha\beta\gamma\delta} = 0$ ← Characterizes Minkowski!

Ricci tensor: R_{ij} or $Ric_{ij} = R^{\mu}{}_{i\mu j} = g^{\mu\nu} R_{\mu i \nu j}$

Ricci scalar (or scalar curvature): $R = g^{ij} R_{ij}$.

Einstein tensor: $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$

Using the Bianchi identities: We can show that G is always divergence free, i.e. $g^{\alpha\beta} \nabla_\alpha G_{\beta i} = \nabla^a G_{ai} = 0$

(Exercise)

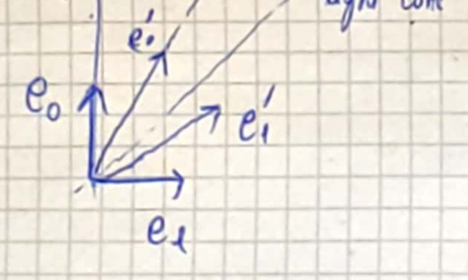
Special Relativity

Formulated by Einstein in 1905. Geometric reformulation: Minkowski 1908

Space and time: $(\mathbb{R}^{3+1}, \eta)$

Inertial observers: Move in timelike straight lines

Let's restrict to $\mathbb{R}^{1+1}$ for simplicity. In the orthonormal frame of reference $\{e_0, e_1\}$ of an inertial observer γ:



If γ' a different inertial observer: They have a different associated orthonormal frame.

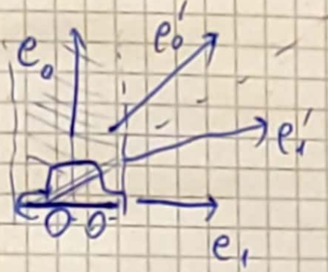
$\{e_0, e_1\}$ is related to $\{e_0', e_1'\}$ by a <u>linear isometry</u> of $(\mathbb{R}^{1+1}, \eta)$.

Note: light "moves" at the same speed for both.

The sets of "constant time" for γ: lines orthogonal to e_0
(so $\parallel$ to e_1)

So γ and γ' have different notion of "simultaneous" events!

So: length of a bus:



Seems to "shrink" for observers in relative motion

Proper time of an observer: Length of timelike curve
(invariant under changes of frame)

Physical quantities: <u>Invariant under changes of inertial frame</u>

<u>Twin paradox</u>: ...

<u>Overall</u>: Special relativity is the study of the geometry of $(\mathbb{R}^{3+1}, \eta)$
inertial observers $\leftrightarrow$ timelike geodesics & orthonormal frames.

General relativity:

Incorporation of gravity into special relativity.

Einstein quickly realized that it had to be based on two principles:

① Principle of equivalence

Free-falling observers experience the same physical laws as inertial ones.

② Principle of general covariance

Physical laws must be "invariant" under change of "frame".

Unifying setup: Spacetime is a Lorentzian manifold (M^{3+1}, g)

g: Incorporates the properties of gravity

- Local frame $\leftrightarrow$ local coordinate system
- Inertial / free falling observers: Move along timelike geodesics

① $\Rightarrow$ In normal coordinates: $g \approx \eta$ to top 2 orders

②: Physical laws must be "geometric", i.e. "coordinate independent"

How does matter/energy tell spacetime how to curve?

Newtonian gravity: $t \longrightarrow (x^1(t), x^2(t), x^3(t))$ free-falling observer

$$\ddot{x} = -\nabla\phi \longrightarrow \text{gravitational potential}$$

Poisson eqn: $\Delta_{\mathbb{R}^3}\phi = 4\pi\rho \leftarrow$ mass density